Granularity Adjustment in a General Factor Model

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Abstract

The granularity adjustment technique is embedded into a general multi-factor model. This allows a very simple statement of the conditions under which the impact of unsystematic risk factors asymptotically vanishes. It has always been taken for granted that the granularity adjustment must be positive. In this paper, a counter-example with negative value of the granularity adjustment is given for the well-known Vasicek (2002) model. This means a discount in terms of capital reserves for a less diversified credit portfolio. An in-depth analyses of the analytical formula of the granularity adjustment reveals that such negative values are possible if the conditional variance is higher for less favourable values of the systematic risk factors. The reason is that this implies a relatively high survival probability in bad states of nature.

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1. Introduction

From the classical capital asset pricing model, the distinction between systematic and unsystematic risk is well known. Unlike systematic risk, unsystematic or idiosyncratic risk can be eliminated through diversification. Exactly the same also applies to a credit portfolio, where the influence of borrower specific risk vanishes completely in a sufficiently diversified portfolio. This can formally be shown by a simple application of the law of large numbers to a general factor model. Because the influence of an individual borrower on a large credit portfolio is very small, the aggregated loss will become non-stochastic if the values of the systematic risk factors are taken as given. The loss then equals the conditional mean of the aggregated loss, which depends only on the more or less favourable realization of the systematic risk factors.

However, no real-world credit portfolio is perfectly diversified. In calculating Value at Risk (VaR) for a credit portfolio, a correction has therefore to be made in order to account for the remaining unsystematic risk. The so-called granularity adjustment technique was introduced by Gordy (2003), (2004). A closed-form expression for the adjustment has been developed by Wilde (2001), Martin and Wilde (2002) and Emmer/Tasche (2005). The granularity adjustment was also part of an earlier version of the new capital accord (Basel II, see BCBS (2004)). Although the granularity adjustment had been later dropped from the final version of the capital accord, the concept is nonetheless interesting from a theoretical point of view. The key idea is that instead of calculating both types of risks simultaneously, a two step model can be developed with an add-up for unsystematic risk.

In this paper, the granularity adjustment technique is embedded into a general multi-factor model. This allows a very simple statement of the conditions under which the impact of unsystematic risk factors asymptotically vanishes. In the existing literature, the analytical formula for the granularity adjustment has only been stated for the one-factor model. Pykhtin (2004) provided a different application of the granularity adjustment technique to the multi-factor case. He develops an approximation to a multi-factor model via a comparable one-factor model. With increasing factor correlation, the respective adjustment term converges to zero.

The granularity adjustment has mostly been considered as a technical tool which could be useful in calculating capital requirements. But it may also provide theoretical insights. For example, it has always been taken for granted that the granularity adjustment must be positive. In this paper, a counter-example with negative value of the granularity adjustment is given for the well-known

Vasicek (2002) model. This means a discount in terms of capital reserves for a less diversified credit portfolio. An in-depth analyses of the analytical formula of the granularity adjustment reveals that such negative values are possible if the conditional variance is higher for less favourable values of the systematic risk factors. The reason is that this implies a relatively high survival probability in bad states of nature.

The paper is organized as follows: Section 2 develops the general factor model and describes its asymptotic behaviour. Section 3 is devoted to the derivation and analysis of the granularity adjustment. An example where the granularity adjustment is negative is given together with a theoretical analyses of this phenomena. Some final remarks are given in Section 4.

2. General Factor Model

2.1 Basic Assumptions

Consider a portfolio of n loans with exposure sizes $A_1, .., A_n$. As a percentage of exposure size, the difference between the current value of each loan and the value at the end of the planning horizon (e.g. one year) is described by a random loss variable L_i. Formally, the relative loss L_i of the value of the loan could be positive as well as negative. It is therefore irrelevant whether losses are defined on a book-value or a mark-to-market basis. For example, if a mark-to-market model is used, an upgrading will result in a gain in market value and consequently implies a negative value of the loss variable L_i.

Let each $L_i = L_i(X, \varepsilon_i)$ be given as a function of some systematic risk factors $X = (X_1, \ldots, X_k)$ and an unsystematic risk factor ε_i. The systematic risk factors may also be called background factors and reflect the state of the business cycle in the different industry sectors. Each systematic risk factor can be thought of being assigned to a certain sector of the economy. The systematic risk factors generally have an influence on more than one borrower in the portfolio and are the reason why default events are stochastic dependent. On the other hand, each unsystematic risk factor ε_i has an influence on only one specific borrower. Unlike for the systematic risk factors, which may or may not be correlated, unsystematic risk factors are always assumed to be pairwise independent.

Many credit risk models can be seen as special cases of this simple but very general approach. Structural models such as the Merton (1974) model or CreditMetrics (1997) assume that default events or rating changes are driven by the evolution of the value of the firm assets, which in turn depend on the realization of some systematic and unsystematic risk factors. The risk factors therefore indirectly determine the potential loss $L_i=L_i(X,\varepsilon_i)$ of each loan. Of course, the concrete functional relationship depends on how the particular model is specified, which however is not relevant for the general analysis.

A well-known example for an intensity or default rate model is CreditRisk+ (1997). This model assumes that default probabilities $p_i = p_i(X)$ are not constant, but a function of certain background factors $X = (X_1,\ldots,X_k)$. In order to match this into the above framework, assume that to each borrower there is assigned an additional unsystematic risk factor ε_i and then define:

$$L_i(X,\varepsilon_i)=\begin{cases} LGD_i, & if\ \varepsilon_i<N^{-1}[p_i(X)] \\ 0 & otherwise \end{cases} \quad (1)$$

Here, it is assumed that the ε_i are standard normal distributed and N^{-1} is the inverse of the cumulative normal distribution function. LGD_i is the loss given default which will arise with probability $p_i = p_i(X)$.

2.2 Diversification

Having the general factor model stated, it is now possible to clarify the role of diversification. As a percentage of total exposure, the random loss of the entire portfolio at the end of the risk horizon is

$$L_P=\frac{\sum_{i=1}^{n} A_i L_i}{\sum_{i=1}^{n} A_i} \quad (2)$$

Now assume that the realizations of the systematic risk factors $X = (X_1,\ldots,X_k)$ occur before the realizations of the unsystematic risk factors ε_i. With given values of the systematic risk factors, L_P is the sum of stochastically independent random variables. Thus, the central limit theorem can be applied. Conditional on X, the portfolio loss variable L_P is asymptotically normal-distributed with mean

$$\mu(L_P|X)=\frac{\sum_{i=1}^{n} A_i \mu(L_i|X)}{\sum_{i=1}^{n} A_i} \tag{3}$$

and variance

$$\sigma^2(L_P|X)=\frac{\sum_{i=1}^{n} A_i^2 \sigma^2(L_i|X)}{(\sum_{i=1}^{n} A_i)^2} \tag{4}$$

However, it is easy to show that if $0<A_{min}<A_i<A_{max}$ and $\sigma^2(L_i|X)<\sigma^2_{max}$ for all i with finite boundaries A_{max} and σ^2_{max}, then $\sigma^2(L_P|X)\to 0$ as $n\to\infty$ for every given realization of X. For n sufficiently large, the conditional variance tends to zero and the probability for an arbitrary small deviation of L_P from the conditional mean $\mu(L_P|X)$ gets arbitrary small.

Therefore, as a consequence of the law of large numbers, the conditional portfolio loss becomes non-stochastic in a very large, infinitely fine-grained credit portfolio. This is the mathematical formulation of the fact how borrower-specific or unsystematic risk can be eliminated through diversification. The only risk that remains is systematic risk, that is the risk that the actual values of the systematic risk factors $X = (X_1,...,X_k)$ result in a higher or lower value of the conditional mean $\mu(L_P|X)$. If systematic risk factors are varying, the portfolio loss, considered as a percentage of total exposure, fluctuates respectively.

If some lumpy credit risk remains within the portfolio, the then non-zero conditional variance $\sigma^2(L_P|X)$ is a natural measure for the amount of unsystematic risk inherent to the credit portfolio. The conditional variance will therefore play an prominent role in the formula for the granularity adjustment to be developed later. Note that the conditional variance $\sigma^2(L_P|X)$ depends on the realization of the systematic risk factors. In the given context, the values of $\sigma^2(L_P|X)$ in those scenarios where the realization of the systematic risk factors give rise to high losses are of particular importance.

Two additional remarks concerning the conditional variance can be made. First, as a direct consequence of the so-called law of conditional variance, the average conditional variance over all possible scenarios for the systematic risk factors equals the difference between the variance of L_P and the variance of $\mu(L_P|X)$:

$$\mu[\sigma^2(L_P|X)] = \sigma^2(L_P) - \sigma^2[\mu(L_P|X)] \quad (5)$$

That is, the expectation of $\sigma^2(L_P|X)$ is that part of the portfolio variance that is caused by unsystematic risk.

Second, the similarities between the conditional variance and the Herfindahl index are obvious. The Herfindahl index

$$H = \frac{\sum_{i=1}^{n} A_i^2}{(\sum_{i=1}^{n} A_i)^2} \quad (6)$$

is an often used measure to quantify the degree of concentration in credit portfolios. It is proportional to conditional variance $\sigma^2(L_P|X)$ if it is assumed that for each borrower i, the conditional variances $\sigma^2(L_i|X)$ of the individual loan loss variables L_i are the same. This implies that differences regarding the distribution of potential losses between the different borrowers can be neglected. Concentration risks can then only arise from differences regarding the exposure sizes A_i. However, if loans not only differ with respect to exposure sizes, but also with respect to e.g. default probabilities or losses given default, then the Herfindahl index might be a to simple measure of concentration risks.

2.3 One-factor model

As an illustration, consider a one factor model based on the following assumptions:

1) the loss variable L_i is a decreasing function of only one systematic risk factor X, i.e. X is a scalar

2) unsystematic risk is perfectly diversified away, i.e. $L_p = \mu(L_P \mid X)$

In this case, an explicit expression for portfolio VaR with confidence level $1\text{-}\alpha$ can be given. Because of

$$\alpha = Prob[L_P > VaR_{1-\alpha}(L_P)] = Prob[\,\mu(L_P \mid X) > VaR_{1-\alpha}(L_P)\,] = Prob(X < x_\alpha) \quad (7)$$

where x_α is the respective quantile of the systematic risk factor *X,* portfolio *VaR*, considered as a percentage of total exposure, is given as:

$$VaR_{1-\alpha}(L_P) = \mu(L_i| X=x_\alpha) = \frac{\sum_{i=1}^{n} A_i \mu(L_i|X=x_\alpha)}{\sum_{i=1}^{n} A_i} \tag{8}$$

A direct decomposition of portfolio *VaR* is obviously possible: If the bank wants to survive with a probability of at least $1\text{-}\alpha$, the amount of capital that must be reserved for each Euro borrowed to borrower *i* is exactly given as $\mu(L_i| X=x_\alpha)$. Marginal *VaR* in a one factor model is then given as the expected loss conditional on $X=x_\alpha$..

A special version of the one-factor model is attributed to Vasicek (2002). It assumes that borrower *i* defaults if the return of the firms assets falls below a certain threshold D_i:

$$r_i = \sqrt{\rho}\, X + \sqrt{1-\rho}\, \varepsilon_i < D_i \tag{9}$$

Here, ρ is the correlation coefficient of the asset returns and X, ε_i are independent standard normal distributed random variables with mean zero and variance one. The coefficients are chosen so that r_i is also standard normal distributed. The relationship between default threshold and probability of default is $PD_i=N^{-1}(D_i)$, where N is the cumulative standard normal distribution function. With default resulting in a loss given default LGD_i (as a percentage of the exposure A_i), one gets the following formula for marginal *VaR*:

$$\begin{aligned} \mu(L_i| X=x_\alpha) &= LGD_i\, Prob\left(\varepsilon_i < \frac{ND^{-1}(PD_i) - \sqrt{\rho}\, x_\alpha}{\sqrt{1-\rho}}\right) \\ &= LGD_i\, N\left(\frac{ND^{-1}(PD_i) - \sqrt{\rho}\, x_\alpha}{\sqrt{1-\rho}}\right) \end{aligned} \tag{10}$$

This type of model has also been adopted by the Basel Committee for Banking Supervision in its proposals for the new Capital Accord. For example, in the consultative paper from January 2001, formula (12) with $x_{0,5\%}$ = -2,57 and ρ = 0,2 for a corporate loan portfolio was used to calculate the capital charge for a loan with default probability PD_i. *S*ome modifications have been made in the final version of the Capital Accord, but the main idea was preserved.

The assumption of only one systematic risk factor has been made by the Basel Committee because this is the only case where capital charges are independent from portfolio composition. This is a problematic assumption because it assumes a completely parallel development of business cycles in all countries and industries. It is therefore irrelevant whether all borrowers of the bank belong to the same sector or not. Contrary to that, exposure to systematic risk can be reduced in reality if loans are well distributed over different industry and country sectors. But, unlike unsystematic risk, exposure to systematic risk does nether completely vanish even in a perfectly diversified portfolio.

In a multi-factor model, an exact decomposition of *VaR* invariant from portfolio composition is no longer possible. However, as it has been shown that the conditional mean is generally the derivative of *VaR*,[1] the conditional mean remains to be a first order approximation of *VaR* contributions also in a multi-factor model. The expectation is then to be calculated conditional on $L_P = VaR_{1-\alpha}(L_P)$. Because this can no longer be simplified to $X = x_\alpha$ if X is not a scalar*,* marginal *VaR* is no longer portfolio invariant. Marginal capital requirements for an additional loan e.g. to a tech firm will then not only depend on the individual default probability, but regularly also on the overall exposure of the existing credit portfolio to the tech sector.

3 Granularity Adjustment

3.1 Formula for Granularity Adjustment

It has been shown that in a perfectly diversified credit loan portfolio, the random variable L_P can be replaced by the random variable $\mu(L_P|X)$. If the portfolio is not perfectly diversified, an adjustment for unsystematic risk has to be made. The so called granularity adjustment is the difference of *VaR* for $\mu(L_P|X)$ and *VaR* for L_P.

The granularity adjustment can be calculated via a sensitivity analysis of *VaR*. However, because the first derivative of *VaR* equals the conditional mean of the marginal risk, the first order approximation of the error term in such a sensitivity analysis is zero. It is therefore necessary to also know the second derivative of *VaR*. With the technical details left to appendix A, one finally gets

[1] In the appendix, it is shown that $VaR(Y+hZ) \approx VaR(Y) + h\,\mu[Z|Y=VaR(Y)]$ if $h \approx 0$ for continuous distributed random variables Y and Z. See also Gourieroux, Laurent & Scaillet (2000).

the following closed-end formula:

$$
\begin{aligned}
&VaR_{1-\alpha}(L_P)\\
&\qquad = VaR_{1-\alpha}[\mu(L_P|X)+L_P-\mu(L_P|X)]\\
&\qquad \approx VaR_{1-\alpha}[\mu(L_P|X)]\\
&\qquad\qquad -\frac{1}{2}\left[\frac{\delta\sigma^2[L_P|\mu(L_P|X)=s]}{\delta s}+\sigma^2[L_P|\mu(L_P|X)=s]\frac{\delta ln\, f_{\mu(L_P|X)}(s)}{\delta s}\right]_{s=VaR_{1-\alpha}[\mu(L_P|X)]}
\end{aligned}
\tag{11}
$$

Here, $f_{\mu(L_P|X)}(s)$ denotes the density of the random variable $\mu(L_P|X)$. Note that $\mu(L_P|X)$ is a scalar defined as a function of one or more systematic risk factors $X = (X_1,\dots,X_k)$. Contrary to the results presented in the literature, this formula for the granularity adjustment is not restricted to the one factor case.

An illustration with a very simple example may be useful. Consider a one-factor model for a completely homogeneous credit portfolio with A_i=1 for all i and

$$
L_i = \begin{cases} 1 & with\ probability\ p(X) \\ 0 & otherwise \end{cases} \tag{12}
$$

where $p(X)$ is a monotone decreasing function of the systematic risk factor X. Then:

$$
\begin{aligned}
&\mu(L_P|X)=p(X)\\
&\sigma^2[L_P|\mu(L_P|X)=s]=\frac{s(1-s)}{n}
\end{aligned}
\tag{13}
$$

Formula (11) then simplifies to

$$
VaR(L_P)\approx p(x_\alpha)-\frac{1-2\,p(x_\alpha)}{2\,n}-\frac{p(x_\alpha)(1-p(x_\alpha))}{2\,n}\frac{\delta ln\, f_{p(X)}(s)}{\delta s}\Big|_{s=p(x_\alpha)} \tag{14}
$$

In this case, the granularity adjustment is inversely proportional to the number of loans n and converges to zero as $n \to \infty$.

3.2 Sign of the granularity adjustment

It has always been taken for granted in the existing literature that the granularity adjustment is positive. However, if ones looks to the analytical formula for the granularity adjustment given by equation (11), it is not immediately clear whether this is indeed always the case.

In order to develop a counterexample, consider the Vasicek model presented above in chapter 2.3. For a completely homogeneous loan portfolio, the conditional default probability is:

$$p(X)=N\left(\frac{N^{-1}(PD)-\sqrt{\rho}\,X}{\sqrt{1-\rho}}\right) \tag{15}$$

In appendix B, it is shown that this implies

$$\frac{\delta \ln f_{p(X)}(s)}{\delta s}\Big|_{s=p(x_\alpha)} = \frac{N^{-1}[p(x_\alpha)](2\rho-1)+N^{-1}(PD)\sqrt{1-\rho}}{\rho \; n[N^{-1}(p(x_\alpha))]} \tag{16}$$

where *n()* denotes the density of the standard normal distribution. With α=30%*,* *PD*=20% and ρ=0,95, one has $p(x_\alpha) = p(-0{,}52400) = 0{,}06957$ and equations (14) then equals

$$VaR_{70\%}(L_P) \approx 0{,}06957-\frac{0{,}04311}{n} \tag{17}$$

If all loans have default a probability of 20% and the bank wants to survive with 70% probability, the capital charge in a perfectly diversified portfolio would be 6,957%. However, in this example, a not perfectly diversified loan portfolio requires a slightly lower (!) capital charge. Although the given choice of the parameters may not be very realistic[2], the example shows that a negative granularity adjustment is indeed possible, at least theoretically.

[2] Less realistic seem not to exit, as an intensive numerical analysis has shown. Note that in this example, the 20% default probability of the loans is lower than the target 30% survival probability of the bank. However, if more and more loans are added to the portfolio, almost certainly some of these loans will default and some capital reserves are required to cover these losses.

From the analytical formula for the granularity adjustment, an explanation is possible how the lack of diversification could, in certain cases, result in a lower *VaR*. First note that with perfect diversification, the bank collapses if $\mu(L_P|X) > VaR_{1-\alpha}(L_P)$ and survives if $\mu(L_P|X) < VaR_{1-\alpha}(L_P)$. If the credit loan portfolio is not perfectly diversified, the bank could also collapse if $\mu(L_P|X) < VaR_{1-\alpha}(L_P)$, and an additional capital buffer is therefore necessary to cover unsystematic risk. However, one should also note that for a not perfectly diversified bank it is also possible to survive even though $\mu(L_P|X) > VaR_{1-\alpha}(L_P)$. In the later case, in which all perfectly diversified banks would collapse, the lack of diversification is obviously an advantage.

Which of these two cases has greater impact depends on the amount of unsystematic risk in the respective scenarios, which is expressed by the value of the conditional variance, and also the occurrence probabilities of these scenarios. If the conditional variance $\sigma^2(L_P|\mu(L_P|X)=s)$ is an increasing function of s, the probability that the bank survives even though the realization of the systematic factors is such that $\mu(L_P|X) > VaR$ is relatively higher than the risk of collapse given a scenario with $\mu(L_P|X) < VaR$. In this case, the first summand $-\delta\sigma^2/\delta s$ within the formula for the granularity adjustment is negative.

The second summand $-(1/2)\sigma^2\, \delta ln(f_\mu)/\delta s$ of the granularity adjustment is positive if the density of the random variable $\mu=\mu(L_P|X)$ slopes downwards in the right tail, which will usually be the case. The occurrence probability of a scenario where the conditional mean is above *VaR* - in which case all perfectly diversified banks would survive - is then higher than the probability of the opposite. However, as the above example shows, there are certain cases where a negative first summand within the granularity adjustment outweighs a positive second summand.

4. Conclusion

The analytical formula for the granularity adjustment has been embedded into a general multi-factor model. This multi-factor model allows to distinguish between the unconditional and the conditional world. In the later case, in which values of the systematic risk factors are taken as given, the aggregated portfolio loss is the sum of stochastic independent random variables and therefore converges, if considered as a percentage of overall exposure, to the respective conditional mean. However, only in a one-factor model this can be exploit for a decomposition of the portfolio loss. Although the conditional mean is the general expression for marginal *VaR*, in a multi-factor framework it depends on the overall composition of the portfolio which realizations of the systematic risk factors are particularly bad. The condition for which the conditional mean is to be calculated can then not be stated independent from the portfolio.

In addition to the conditional mean, the conditional variance of the portfolio loss is also a useful variable in analyzing the riskiness of the portfolio. It is given as the weighted average of the conditional variances of the individual loans, with the weights depending on exposure sizes. If all these individual loans have the same conditional variance, the conditional variance of the portfolio would be proportional to the Herfindahl index. But if loans do not only have different exposure sizes, but also differ with respect to their default probabilities or the amount of loss in the event of a default, the Herfindahl index may be an inappropriate measure of concentration risks.

In general, the conditional variance depends on the value of the systematic risk factors, and its value for particularly bad realizations of these factors are of special interest. A high conditional variance in a very bad state of nature implies a relatively high chance that the actual portfolio loss is lower than its conditional mean. Formally, this is the case then the conditional variance increases together with the conditional mean. If the divergence between actual loss and its conditional mean prevents a collapse of the bank, the lack of diversification would be an advantage. In this paper, it has been shown that this indeed implies the existence of numerical examples with negative granularity adjustment.

The possibility of a negative granularity adjustment should not be very surprising as Artzner et al. (1999) have shown that *VaR* is not sub-additive and therefore does not always account correctly for diversification. Though it seems that a negative granularity adjustment is a rare event which only occurs for very unusual parameter values, it gives another hint that *VaR* is a problematic measure of risk.

Appendix A: Granularity Adjustment

First note that

$$
\begin{aligned}
VaR_{1-\alpha}(L_P) &= VaR_{1-\alpha}[\mu(L_P|X)+L_P-\mu(L_P|X)] \\
&\approx VaR_{1-\alpha}[\mu(L_P|X)]+\frac{\delta VaR_{1-\alpha}[\mu(L_P|X)+h(L_P-\mu(L_P|X))]}{\delta h}\Big|_{h=0} \\
&\quad +\frac{1}{2}\frac{\delta^2 VaR_{1-\alpha}[\mu(L_P|X)+h(L_P-\mu(L_P|X))]}{\delta h^2}\Big|_{h=0}
\end{aligned}
$$

Formula (11) for the granularity adjustment then is an immediate consequence of the following

Lemma:

$$
\frac{\delta VaR_{1-\alpha}(Y+hZ)}{\delta h}=\mu[Z|Y+hZ=VaR_{1-\alpha}(Y+hZ)]
$$

and

$$
\frac{\delta^2 VaR_{1-\alpha}(Y+hZ)}{\delta h^2}=-\frac{1}{2}\left[\frac{\delta\sigma^2(Z|Y+hZ=s)}{\delta s}+\sigma^2(Z|Y+hZ=s)\frac{\delta ln\, f_{Y+hZ}(s)}{\delta s}\right]_{s=VaR_{1-\alpha}(Y+hZ)}
$$

Proof:

With abbreviation $VaR=VaR_{1-\alpha}(Y+hZ)$ one has

$$
\begin{aligned}
0 &= \frac{\delta}{\delta h}Prob(Y+hZ>VaR) \\
&= \frac{\delta}{\delta h}\int_{-\infty}^{\infty}\int_{VaR-hz}^{\infty} f(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\left(\frac{\delta VaR}{\delta h}-z\right) f(VaR-hz,z)\,dz
\end{aligned}
$$

Because of

$$f(VaR-hz,z) \;=\; f_Z(z|Y+hZ=VaR) \;\; f_{Y+hZ}(VaR)$$

the result for the first derivative of VaR follows by dividing through by $f_{Y+hZ}(VaR)$. To get the second derivative, one proceeds as follows:

$$\begin{aligned}
0 = \;& \frac{\delta}{\delta h}\int_{-\infty}^{\infty}\left(\frac{\delta VaR}{\delta h}-z\right) f(VaR-hz,z)\,dz \\
= \;& \int_{-\infty}^{\infty} \frac{\delta^2 VaR}{\delta^2 h} f(VaR-hz,z)+\left(\frac{\delta VaR}{\delta h}-z\right)\frac{\delta f(VaR-hz,z)}{\delta h}\,dz \\
= \;& \int_{-\infty}^{\infty} \frac{\delta^2 VaR}{\delta^2 h} f(VaR-hz,z)+\left(\frac{\delta VaR}{\delta h}-z\right)^2 \; \frac{\delta f(s-hz,z)}{\delta s}\Big|_{s=VaR}\,dz \\
= \;& \frac{\delta^2 VaR}{\delta^2 h} f_{Y+hZ}(VaR)+\int_{-\infty}^{\infty}\left(\frac{\delta VaR}{\delta h}-z\right)^2 \; \frac{\delta[\,f_Z(z|Y+hZ=s) \;\; f_{Y+hZ}(s)]}{\delta s}\Big|_{s=VaR}\,dz \\
= \;& \frac{\delta^2 VaR}{\delta^2 h} f_{Y+hZ}(VaR)+\frac{\delta}{\delta s}\mu\left[\left(\frac{\delta VaR}{\delta h}\right)^2-2\frac{\delta VaR}{\delta h}z+z^2|Y+hZ=s\right]_{s=VaR} \; f_{Y+hZ}(VaR) \\
& \qquad + \;\; \mu\left[\left(\frac{\delta VaR}{\delta h}-z\right)^2|Y+hZ=VaR\right] \;\; \frac{\delta f_{Y+hZ}(s)}{\delta s}\Big|_{s=VaR} \\
= \;& \frac{\delta^2 VaR}{\delta^2 h} f_{Y+hZ}(VaR)+\frac{\delta[0-\mu^2(Z|Y+hZ=s)+\mu(Z^2|Y+hZ=s)]}{\delta s}\Big|_{s=VaR} \;\; f_{Y+hZ}(VaR) \\
& \qquad + \;\; \mu[(\mu(Z|Y+hZ=VaR)-z)^2|Y+hZ=VaR] \;\; \frac{\delta f_{Y+hZ}(s)}{\delta s}\Big|_{s=VaR} \\
= \;& f_{Y+hZ}(VaR)\;\left[\frac{\delta^2 VaR}{\delta^2 h}+\left(\frac{\delta\sigma^2(Z|Y+hZ=s)}{\delta s}+\sigma^2(Z|Y+hZ=VaR)\frac{\delta lnf_{Y+hZ}(s)}{\delta s}\right)_{s=VaR}\;\right]
\end{aligned}$$

q.e.d.

Appendix B

If X is a standard normal random variable and $f_{p(X)}(s)$ denotes the density of

$$p(X)=N(\frac{N^{-1}(PD)-\sqrt{\rho}X}{\sqrt{1-\rho}})$$

Then:

$$\frac{\delta ln\, f_{p(X)}(s)}{\delta s}\Big|_{s=p(x_\alpha)} = \frac{N^{-1}[p(x_\alpha)](2\rho-1)+N^{-1}(PD)\sqrt{1-\rho}}{\rho\;\; n[N^{-1}(p(x_\alpha))]}$$

Proof:

$$\begin{aligned} Prob(p(X)<s) &= Prob[N(\frac{D-\sqrt{\rho}X}{\sqrt{1-\rho}})<s] \\ &= Prob[X>\frac{D-N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}}] \\ &= 1-Prob[X<\frac{D-N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}}] \\ &= 1-N[\frac{D-N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}}] \end{aligned}$$

with $D=N^{-1}(PD)$. The density of $p(X)$ is then given as:

$$f_{p(X)}(s)=\frac{d}{ds}Prob(p(X)<s)=n[\frac{D-N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}}]\frac{\sqrt{1-\rho}}{n(N^{-1}(s))\sqrt{\rho}}$$

It follows that

$$\ln(f_{p(X)}(s)) = \frac{-1}{2}\left[\frac{D - N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}}\right]^2 + \frac{(N^{-1}(s))^2}{2} + \ln\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)$$

and:

$$\frac{\delta ln f_{p(X)}(s)}{\delta s} = \frac{D - N^{-1}(s)\sqrt{1-\rho}}{\sqrt{\rho}} \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \frac{1}{n(N^{-1}(s))} + \frac{N^{-1}(s)}{n(N^{-1}(s))}$$

$$= \frac{N^{-1}(s)(2\rho-1) + D\sqrt{1-\rho}}{\rho} \frac{1}{n(N^{-1}(s))}$$

q.e.d.

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